

# On a weighted variable spaces $L_{p(x),\omega}$ for $0 < p(x) < 1$ and weighted Hardy inequality

ROVSHAN A.BANDALIEV

**ABSTRACT.** In this paper a weighted variable exponent Lebesgue spaces  $L_{p(x),\omega}$  for  $0 < p(x) < 1$  is investigated. We show that this spaces is a quasi-Banach spaces. Note that embedding theorem between weight variable Lebesgue spaces is proved. In particular, we show that  $L_{p(x),\omega}(\Omega)$  for  $0 < p(x) < 1$  isn't locally convex. Also, in this paper a some two-weight estimates for Hardy operator are proved.

*Keywords and phrases:* Variable Lebesgue space, weights, quasi-Banach space, topology, embedding, Hardy operator.

2000 *Mathematics Subject Classifications:* Primary 46B50, 47B38; Secondary 26D15.

## 1. Introduction.

It is well known that the variable exponent Lebesgue space  $L_{p(x)}$  for  $p(x) \geq 1$  appeared in the literature for the first time already in [13]. Further development of this theory was connected with the theory of modular function spaces. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [12]). The next step in the investigation of variable exponent spaces was given in [16] and in [8]. But the variable exponent Lebesgue space for  $0 < p(x) < 1$  very less studied. Note that the space  $L_{p(x)}$  for  $0 < p(x) < 1$  isn't modular function spaces. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [14], [17],[18]). For detailed information about variable exponent Lebesgue space  $L_{p(x)}$  for  $p(x) \geq 1$  we refer to [7].

Let  $R^n$  be the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$  and  $\Omega$  be a Lebesgue measurable subset in  $R^n$  and  $|x| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}$ . Suppose that  $p$  is a Lebesgue measurable function on  $\Omega$  such that  $0 < \underline{p} \leq p(x) \leq \bar{p} < 1$ ,  $\underline{p} = \text{ess inf}_{x \in \Omega} p(x)$ ,  $\bar{p} = \text{ess sup}_{x \in \Omega} p(x)$ , and  $\omega$  is a weight function on  $\Omega$ , i.e.  $\omega$  is non-negative, almost everywhere (a.e.) positive function on  $\Omega$ . The Lebesgue measure of a set  $\Omega$  will be denoted by  $|\Omega|$ . It is well known that  $|B(0, 1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}$ , where  $B(0, 1) = \{x : x \in R^n; |x| < 1\}$ . Further, in this paper all sets and functions are supposed Lebesgue measurable.

## 2. Preliminaries

**Definition 1.** By  $L_{p(x),\omega}(\Omega)$  we denote the set of measurable functions  $f$  on  $\Omega$  such that

$$I_{p,\omega}(f) = \int_{\Omega} (|f(x)| \omega(x))^{p(x)} dx < \infty.$$

Note that the expression

$$\|f\|_{L_{p(x),\omega}(\Omega)} = \|f\|_{p,\omega,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\lambda} \right)^{p(x)} dx \leq 1 \right\} \quad (2.1)$$

defines a quasi-Banach spaces.

We note some main properties of this spaces.

1) For every  $0 < \|f\|_{p,\omega,\Omega} < \infty$ ,  $I_{p,\omega} \left( \frac{f}{\|f\|_{p,\omega,\Omega}} \right) = 1$ .

If  $I_{p,\omega} \left( \frac{f}{\|f\|_{p,\omega,\Omega}} \right) < 1$ , we can find  $0 < \lambda \leq \|f\|_{p,\omega,\Omega}$  such that  $I_{p,\omega} \left( \frac{f}{\lambda} \right) < 1$ . Indeed, let  $\lambda = \|f\|_{p,\omega,\Omega} I_{p,\omega}^{1/\bar{p}} \left( \frac{f}{\|f\|_{p,\omega,\Omega}} \right)$ . Then  $\lambda < \|f\|_{p,\omega,\Omega}$  and the inequality

$$\begin{aligned} I_{p,\omega} \left( \frac{f}{\lambda} \right) &= \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega} I_{p,\omega}^{1/\bar{p}} \left( \frac{f}{\|f\|_{p,\omega,\Omega}} \right)} \right)^{p(x)} dx \\ &\leq I_{p,\omega}^{-1} \left( \frac{f}{\|f\|_{p,\omega,\Omega}} \right) \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx = 1 \end{aligned}$$

is valid. The obtained inequality contradicts to (2.1).

**Remark 1.** Note that property 1) for non-weighted case was proved in [15].

2)  $\min \{ \|f\|_{p,\omega,\Omega}^p, \|f\|_{p,\omega,\Omega}^{\bar{p}} \} \leq I_{p,\omega}(f) \leq \max \{ \|f\|_{p,\omega,\Omega}^p, \|f\|_{p,\omega,\Omega}^{\bar{p}} \}$ .

Let  $\|f\|_{p,\omega,\Omega} \leq 1$ . Using the property 1) we have

$$I_{p,\omega}(f) = \int_{\Omega} \|f\|_{p,\omega,\Omega}^{p(x)} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx \leq \|f\|_{p,\omega,\Omega}^p \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx = \|f\|_{p,\omega,\Omega}^p.$$

Conversely,  $I_{p,\omega}(f) \geq \|f\|_{p,\omega,\Omega}^{\bar{p}}$ . Analogously, is consider the case  $\|f\|_{p,\omega,\Omega} \geq 1$ .

3) The space  $L_{p(x),\omega}(\Omega)$  is real linear spaces.

By using of the property 1), we have

$$\begin{aligned}
& \int_{\Omega} \left( \frac{|f(x) + g(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} dx \\
& \leq \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} dx + \int_{\Omega} \left( \frac{|g(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} dx \\
& \leq \int_{\Omega} 2^{-\frac{p(x)}{p}} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx + \int_{\Omega} 2^{-\frac{p(x)}{p}} \left( \frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx \\
& \leq \frac{1}{2} \left( \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx + \int_{\Omega} \left( \frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx \right) = 1.
\end{aligned}$$

Thus by Definition 1  $\|f + g\|_{p,\omega,\Omega} \leq 2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})$ . Therefore  $f + g \in L_{p(x),\omega}(\Omega)$ .

Let  $\alpha \in R \setminus \{0\}$  and  $f \in L_{p(x),\omega}(\Omega)$ . Now show that  $\alpha f \in L_{p(x),\omega}(\Omega)$ . We get

$$\begin{aligned}
\|\alpha f\|_{p,\omega,\Omega} &= \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|\alpha f(x)| \omega(x)}{\lambda} \right)^{p(x)} dx \leq 1 \right\} \\
&= \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\frac{\lambda}{|\alpha|}} \right)^{p(x)} dx \leq 1 \right\}
\end{aligned}$$

We substitute  $\lambda = |\alpha| \mu$ . Then

$$\begin{aligned}
& \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\frac{\lambda}{|\alpha|}} \right)^{p(x)} dx \leq 1 \right\} \\
&= \inf \left\{ |\alpha| \mu > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\mu} \right)^{p(x)} dx \leq 1 \right\} \\
&= |\alpha| \inf \left\{ \mu > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\mu} \right)^{p(x)} dx \leq 1 \right\} = |\alpha| \|f\|_{p,\omega,\Omega}.
\end{aligned}$$

For  $f = 0$  this fact is trivially. Hence implies that the variable Lebesgue space  $L_{p(x),\omega}(\Omega)$  is real linear space.

4) Let  $\|f\|_{p,\omega,\Omega} = 0$ . Then we proved that  $f = 0$  a.e.  $x \in \Omega$ .

If  $\|f\|_{p,\omega,\Omega} = 0$ , then by (2.1) for all  $\lambda > 0$ ,  $I_{p,\omega} \left( \frac{f}{\lambda} \right) \leq 1$ . For any  $\mu > 0$  and  $\varepsilon \in (0, 1)$ , we have

$$I_{p,\omega} \left( \frac{f}{\mu} \right) = \int_{\Omega} \varepsilon^{p(x)} \left( \frac{|f(x)| \omega(x)}{\varepsilon \mu} \right)^{p(x)} dx \leq \varepsilon^p I_{p,\omega} \left( \frac{f}{\varepsilon \mu} \right) \leq \varepsilon^p.$$

Since  $\varepsilon$  be any number from  $(0, 1)$ , then  $I_{p,\omega} \left( \frac{f}{\mu} \right) = 0$  for all  $\mu > 0$ . Therefore

$$\int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\mu} \right)^{p(x)} dx = 0 \text{ and thus } f = 0 \text{ a.e. } x \in \Omega.$$

5) Let  $|f(x)| \leq |g(x)|$  for a.e.  $x \in \Omega$ . Then  $\|f\|_{p,\omega,\Omega} \leq \|g\|_{p,\omega,\Omega}$ .

Indeed, by virtue of property 1) we have

$$\begin{aligned} \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx &= \int_{\Omega} \left( \frac{|f(x)|}{|g(x)|} \frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx \\ &\leq \int_{\Omega} \left( \frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx = 1. \end{aligned}$$

Thus by Definition 1  $\|f\|_{p,\omega,\Omega} \leq \|g\|_{p,\omega,\Omega}$ .

**Lemma 1.** Let  $0 < \underline{p} \leq p(x) \leq \bar{p} < 1$  and  $f, g \in L_{p(x),\omega}(\Omega)$ . Then

$$\| |f| + |g| \|_{p,\omega,\Omega} \geq \|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}.$$

**Proof.** First we show that the function  $h(t) = t^r$ , for  $0 < r < 1$  and  $t > 0$  is concave. Let  $\alpha + \beta = 1$ , where  $\alpha, \beta \geq 0$ . We proved that  $(\alpha + \beta t)^r \geq \alpha + \beta t^r$ . We consider the function  $F(t) = \frac{(\alpha + \beta t)^r}{\alpha + \beta t^r}$ . Differentiating by  $t$  and after some calculation we have

$$F'(t) = \frac{\alpha \beta p (\alpha + \beta t)^{r-1} (1 - t^{r-1})}{(\alpha + \beta t^r)^2}.$$

Since  $r - 1 < 0$ , then  $t = 1$  is minimal value of the function  $F$  for all  $t > 0$ . Therefore  $F(t) \geq F(1) = 1$ . Thus  $(\alpha + \beta t)^r \geq \alpha + \beta t^r$ . Taking  $t = \frac{t_2}{t_1}$  in last inequality we have  $(\alpha t_1 + \beta t_2)^r \geq \alpha t_1^r + \beta t_2^r$ , i.e. the function  $h(t) = t^r$  is concave.

Now we show a requiring inequality. It is obvious that the case  $f = g = 0$  a.e.  $x \in \Omega$  is trivial. Let  $\|f\|_{p,\omega,\Omega} > 0$  and  $\|g\|_{p,\omega,\Omega} > 0$ . Using concavity property of power function and property 1), we get

$$I_{p,\omega} \left( \frac{|f| + |g|}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \right) = \int_{\Omega} \left( \frac{|f(x)| + |g(x)|}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \omega(x) \right)^{p(x)} dx =$$

$$\begin{aligned}
& \int_{\Omega} \left( \frac{\|f\|_{p,\omega,\Omega} \frac{|f(x)|}{\|f\|_{p,\omega,\Omega}} + \|g\|_{p,\omega,\Omega} \frac{|g(x)|}{\|g\|_{p,\omega,\Omega}}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \omega(x) \right)^{p(x)} dx \\
&= \int_{\Omega} \left( \frac{\|f\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \frac{|f(x)|}{\|f\|_{p,\omega,\Omega}} + \frac{\|g\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \frac{|g(x)|}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} [\omega(x)]^{p(x)} dx \\
&\geq \frac{\|f\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \int_{\Omega} \left( \frac{|f(x)|\omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx + \frac{\|g\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} \int_{\Omega} \left( \frac{|g(x)|\omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} dx \\
&= \frac{\|f\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} + \frac{\|g\|_{p,\omega,\Omega}}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}} = 1.
\end{aligned}$$

Thus  $\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega} \geq \|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}$ . In addition, note that the inequality in the form  $\|f + g\|_{p,\omega,\Omega} \geq \|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}$  doesn't hold for  $L_{p(x),\omega}(\Omega)$ . Indeed, taking  $g = -f$  we can see  $0 \geq 2\|f\|_{p,\omega,\Omega}$ , which is valid only for  $f = 0$  a.e.  $x \in \Omega$ .

This proves the Lemma 1.

**Theorem 1.** Let  $0 < \underline{p} \leq p(x) \leq \bar{p} < 1$  and  $p'(x) = \frac{p(x)}{p(x) - 1}$  and  $\omega$  be a weight function defined on  $\Omega$ . Then the inequality

$$\int_{\Omega} |f(x)g(x)| dx \geq \left( \frac{1}{\bar{p}} + \frac{1}{\bar{p}'} \right) \|f\|_{p,\omega,\Omega} \|g\|_{p',\omega^{-1},\Omega} \quad (2.2)$$

holds for every  $f \in L_{p(x),\omega}(\Omega)$ ,  $g \in L_{p'(x),\omega^{-1}}(\Omega)$  and  $0 < |g(x)| < \infty$ .

**Proof.** We consider the function  $G(t) = \frac{t^s}{s} + \frac{t^{-s'}}{s'}$ , where  $t > 0$ ,  $0 < s = \text{const} < 1$  and  $s' = \frac{s}{s-1}$ . Differentiating by  $t$  we have

$$G'(t) = t^{s-1} - \frac{1}{t^{s'+1}} = \frac{t^{ss'} - 1}{t^{s'+1}},$$

where  $s + s' = ss' < 0$ . Therefore the point  $t = 1$  is maximal value of the function  $G(t)$  for all  $t > 0$ . Thus  $G(t) \leq G(1) = 1$ , i.e.,  $\frac{t^s}{s} + \frac{t^{-s'}}{s'} \leq 1$ . If we take  $t = \frac{a^{1/s'}}{b^{1/s}}$ , then

$$ab \geq \frac{a^s}{s} + \frac{b^{s'}}{s'}, \quad (2.3)$$

where  $a, b > 0$ .

Putting  $a = \frac{|f(x)|\omega(x)}{\|f\|_{p,\omega,\Omega}}$ ,  $b = \frac{|g(x)|\omega^{-1}(x)}{\|g\|_{p',\omega^{-1},\Omega}}$ ,  $s = s(x) = p(x)$ ,  $s' = s'(x) = p'(x)$  in inequality (2.3) and using the property 1) we have

$$\begin{aligned} \int_{\Omega} \frac{|f(x)g(x)|}{\|f\|_{p,\omega,\Omega}\|g\|_{p',\omega^{-1},\Omega}} dx &\geq \int_{\Omega} \frac{1}{p(x)} \left( \frac{|f(x)|\omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx + \int_{\Omega} \frac{1}{p'(x)} \left( \frac{|g(x)|\omega^{-1}(x)}{\|g\|_{p',\omega^{-1},\Omega}} \right)^{p'(x)} dx \\ &\geq \frac{1}{\bar{p}} \int_{\Omega} \left( \frac{|f(x)|\omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} dx + \frac{1}{\bar{p}'} \int_{\Omega} \left( \frac{|g(x)|\omega^{-1}(x)}{\|g\|_{p',\omega^{-1},\Omega}} \right)^{p'(x)} dx = \frac{1}{\bar{p}} + \frac{1}{\bar{p}'}. \end{aligned}$$

Thus the inequality (2.2) is proved.

**Remark 2.** Note that in the proof of Lemma 1, the expression  $\|g\|_{p',\omega^{-1},\Omega}$  was used for negative values of the conjugate function. It should be understood as follows

$$\begin{aligned} \|g\|_{p',\omega^{-1},\Omega} &:= \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{[|g(x)|\omega^{-1}(x)]^{-1}}{\lambda^{-1}} \right)^{-p'(x)} \leq 1 \right\} \\ &= \inf \left\{ \frac{1}{\mu} > 0 : \int_{\Omega} \left( \frac{[|g(x)|\omega^{-1}(x)]^{-1}}{\mu} \right)^{-p'(x)} \leq 1 \right\} = \\ &= \sup \left\{ \mu > 0 : \int_{\Omega} \left( \frac{|g(x)|\omega^{-1}(x)}{\mu} \right)^{p'(x)} \leq 1 \right\}. \end{aligned}$$

**Theorem 2.** Let  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$  and  $r(x) = \frac{p(x)q(x)}{q(x) - p(x)}$ . Suppose that  $\omega_1$  and  $\omega_2$  are weights functions defined in  $\Omega$  and satisfying the condition

$$\left\| \frac{\omega_1}{\omega_2} \right\|_{r,\Omega} < \infty.$$

Then the inequality

$$\|f\|_{p,\omega_1,\Omega} \leq \left( A + B + \|\chi_{\Omega_2}\|_{L_{\infty}(\Omega)} \right)^{1/\underline{p}} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q,\omega_2,\Omega},$$

holds for every  $f \in L_{q(x),\omega_2}(\Omega)$ , where  $\Omega_1 = \{x \in \Omega : p(x) < q(x)\}$ ,  $\Omega_2 = \{x \in \Omega : p(x) = q(x)\}$  and  $A = \sup_{x \in \Omega_1} \frac{p(x)}{q(x)}$ ,  $B = \sup_{x \in \Omega_1} \frac{q(x) - p(x)}{q(x)}$  and  $\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} = \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega_1)} + \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{\infty}(\Omega_2)}$ .

**Proof.** We have

$$\begin{aligned}
\|f\|_{p, \omega_1, \Omega_2} &= \left\| f \omega_2 \frac{\omega_1}{\omega_2} \right\|_{p, \omega_1, \Omega_2} \leq \left\| \frac{\omega_1}{\omega_2} \right\|_{L_\infty(\Omega_2)} \|f \omega_2\|_{p, \Omega_2} \\
&= \left\| \frac{\omega_1}{\omega_2} \right\|_{L_\infty(\Omega_2)} \|f \chi_{\Omega_2}\|_{p, \omega_2, \Omega} \leq \left\| \frac{\omega_1}{\omega_2} \right\|_{L_\infty(\Omega_2)} \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \|f\|_{p, \omega_2, \Omega}.
\end{aligned}$$

Therefore  $\left\| \frac{f}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_\infty(\Omega_2)} \|f\|_{p, \omega_2, \Omega}} \right\|_{p, \omega_1, \Omega_2} \leq \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \leq 1$ . By virtue of property 1)

$$\int_{\Omega_2} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_\infty(\Omega_2)} \|f\|_{p, \omega_2, \Omega}} \right)^{p(x)} dx \leq \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}^p = \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}. \quad (2.4)$$

It is well known that the inequality (2.3) for  $s > 1$  is Young's inequality, i.e.

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'}, \quad (2.5)$$

where  $s' = \frac{s}{s-1}$ . We take  $s = s(x) = \frac{q(x)}{p(x)}$ ,  $a = \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)}$  and  $b = \left[ \frac{\omega_1(x)}{\omega_2(x)} / \left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1} \right]^{p(x)}$ .

Thus  $s' = s'(x) = \frac{q(x)}{q(x) - p(x)}$  and from inequality (2.5), we have

$$\begin{aligned}
\left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1} \|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)} &\leq \frac{p(x)}{q(x)} \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{q(x)} + \frac{q(x) - p(x)}{q(x)} \left[ \frac{\frac{\omega_1(x)}{\omega_2(x)}}{\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1}} \right]^{r(x)} \\
&\leq A \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{q(x)} + B \left[ \frac{\frac{\omega_1(x)}{\omega_2(x)}}{\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1}} \right]^{r(x)}.
\end{aligned}$$

Obviously,  $1 \leq A + B \leq 2$ . Integrating by  $\Omega_1$ , using the property 1), we get

$$\begin{aligned}
&\int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1} \|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)} dx \\
&\leq A \int_{\Omega_1} \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{q(x)} dx + B \int_{\Omega_1} \left[ \frac{\frac{\omega_1(x)}{\omega_2(x)}}{\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega_1}} \right]^{r(x)} dx \leq A + B. \quad (2.6)
\end{aligned}$$

From (2.4) and (2.6) implies that

$$\begin{aligned}
& \int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx = \int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx \\
& + \int_{\Omega_2} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx \leq \int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega_1)} \|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)} dx \\
& + \int_{\Omega_2} \left( \frac{|f(x)| \omega_1(x)}{\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{\infty}(\Omega_2)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx \leq A + B + \|\chi_{\Omega_2}\|_{L_{\infty}(\Omega)}.
\end{aligned}$$

From last inequality we have

$$\begin{aligned}
1 & \geq \int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{\left( A + B + \|\chi_{\Omega_2}\|_{L_{\infty}(\Omega)} \right)^{1/p(x)} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx \\
& \geq \int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{\left( A + B + \|\chi_{\Omega_2}\|_{L_{\infty}(\Omega)} \right)^{1/\underline{p}} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}} \right)^{p(x)} dx.
\end{aligned}$$

Thus

$$\|f\|_{p, \omega_1, \Omega} \leq \left( A + B + \|\chi_{\Omega_2}\|_{L_{\infty}(\Omega)} \right)^{1/\underline{p}} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r(\cdot)}(\Omega)} \|f\|_{q, \omega_2, \Omega}.$$

The theorem is proved.

**Remark 3.** Note that Theorem 2 in the case  $\omega_1 = \omega_2 = 1$  and  $|\Omega| < \infty$  was proved in [15]. In the case  $1 \leq \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < \infty$  for general measures Theorem 2 was proved in [4].

The following theorems are known.

**Theorem 3.** [1] Let  $1 \leq \underline{p} \leq p(x) \leq q(y) \leq \bar{q} < \infty$  for all  $x \in \Omega_1 \subset R^n$  and  $y \in \Omega_2 \subset R^m$ . If  $p(x) \in C(\Omega_1)$ , then the inequality

$$\left\| \|f\|_{L_{p(\cdot)}(\Omega_1)} \right\|_{L_{q(\cdot)}(\Omega_2)} \leq C_{p,q} \left\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \right\|_{L_{p(\cdot)}(\Omega_1)}$$



is valid, where  $C_{p,q} = \left( \|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\bar{p}}{q} - \frac{p}{\bar{q}} \right) (\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty)$ ,  $q = \text{ess} \inf_{\Omega_2} q(x)$ ,  $\bar{q} = \text{ess} \sup_{\Omega_2} q(x)$ ,  $\Delta_1 = \{(x, y) \in \Omega_1 \times \Omega_2 : p(x) = q(y)\}$ ,  $\Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$  and  $C(\Omega_1)$  is the space of continuous functions in  $\Omega_1$  and  $f : \Omega_1 \times \Omega_2 \rightarrow R$  is any measurable function such that

$$\| \|f\|_{q,\Omega_2} \|_{p,\Omega_1} = \inf \left\{ \mu > 0 : \int_{\Omega_1} \left( \frac{\|f(x, \cdot)\|_{q(\cdot),\Omega_2}}{\mu} \right)^{p(x)} dx \leq 1 \right\} < \infty.$$

The following lemmas are known.

**Lemma 2.** [6] Let  $0 < s < 1$ ,  $-\infty < a < b \leq \infty$  and  $f$  is non-negative and decreasing function defined on  $(a, b)$ . Then

$$\left( \int_a^b f(x) dx \right)^s \leq s \int_a^b f^s(x) (x - a)^{s-1} dx.$$

**Lemma 3.** [6] Let  $0 < s < 1$ ,  $-\infty \leq a < b < \infty$  and  $f$  is non-negative and increasing function defined on  $(a, b)$ . Then

$$\left( \int_a^b f(x) dx \right)^s \leq s \int_a^b f^s(x) (b - x)^{s-1} dx.$$

### 3. On a topology of the spaces $L_{p(x),\omega}$ for $0 < p(x) < 1$

Now we formulate some definitions which be characterized of the topology in general vector spaces.

**Definition 2.** A subset  $G$  of a vector space  $X$  is called convex if, for any  $x_1, x_2, \dots, x_m \in G$ ,  $\sum_{i=1}^m \alpha_i x_i \in G$ , where  $\sum_{i=1}^m \alpha_i = 1$  and  $\alpha_i \geq 0$ ,  $i = 1, 2, \dots, m$ . In particular, the subset contains the average  $\frac{1}{m} \sum_{i=1}^m x_i$ .

**Definition 3.** A topological vector space  $X$  is called locally convex if the convex open sets are a base for the topology, i.e., any open set  $U \subset X$  around a point, there is a convex open set  $C$  containing that point such  $C \subset U$ .

We show that the weighted variable Lebesgue spaces  $L_{p(x),\omega}(\Omega)$  isn't locally convex.

**Lemma 4.** *Let  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$  and  $\omega$  be a weight function defined on  $\Omega$  and  $0 < \omega(x) < \infty$  a.e.  $x \in \Omega$ . Then weighted variable Lebesgue spaces  $L_{p(x),\omega}(\Omega)$  isn't locally convex.*

**Proof.** It is obvious that  $\rho(f, g) = \int_{\Omega} [|f(x) - g(x)| \omega(x)]^{p(x)} dx$  is defined a metric on  $L_{p(x),\omega}(\Omega)$ . We consider any open ball neighborhoods 0 :

$$U_R(0) = \{f \in L_{p(x),\omega}(\Omega) : \rho(f, 0) = I_{p(x),\omega}(f) < R\}.$$

We will show that, for any  $\varepsilon > 0$ , the  $\varepsilon$ -ball neighborhoods zero contains functions whose average lies outside the ball of radius  $R$ .

Suppose  $\varepsilon > 0$  and  $m \geq 1$ . We select  $m$  disjoint intervals  $A_1, A_2, \dots, A_m$  in  $\Omega$ , which need not cover of all  $\Omega$ . We put  $f_k = \left(\frac{\varepsilon}{\omega(A_k)}\right)^{1/p(x)} \chi_{A_k}$ , where  $\omega(A_k) = \int_{A_k} [\omega(x)]^{p(x)} dx$  and  $k = 1, 2, \dots, m$ . Then  $I_{p,\omega}(f_k) = \frac{\varepsilon}{\omega(A_k)} \int_{A_k} [\omega(x)]^{p(x)} dx = \varepsilon$ , and so every  $f_k$  is at distance  $\varepsilon$  from 0. But, since the functions  $f_k$  are supported on disjoint sets, their average  $g_m = \frac{1}{m} \sum_{i=1}^m f_i$  satisfies

$$\begin{aligned} I_{p,\omega}(g_m) &= \int_{\Omega} g_m^{p(x)}(x) dx = \int_{\Omega} \frac{1}{m^{p(x)}} \left(\sum_{i=1}^m f_i\right)^{p(x)} [\omega(x)]^{p(x)} dx \\ &\geq \frac{1}{m^{\bar{p}}} \sum_{i=1}^m \int_{\Omega} (f_i(x) \omega(x))^{p(x)} dx = \frac{\varepsilon}{m^{\bar{p}}} \sum_{i=1}^m 1 = m^{1-\bar{p}} \varepsilon. \end{aligned}$$

Then  $I_{p,\omega}(g_m) \rightarrow \infty$ , for  $m \rightarrow \infty$  (depending on  $\varepsilon$ ). Therefore  $\rho(g_m, 0) \rightarrow \infty$ , for  $m \rightarrow \infty$ . Thus the distance between  $g_n$  and 0 can be made as large as desired.

The Lemma 4 is proved.

**Theorem 4.** *Let  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$  and  $\omega$  be a weight function defined on  $\Omega$  and  $0 < \omega(x) < \infty$  a.e.  $x \in \Omega$ . Then  $[L_{p(x),\omega}(\Omega)]^* = \{0\}$ , where  $\star$  - be denoted dual space of  $L_{p(x),\omega}(\Omega)$ , i.e., is the space of continuous linear functionals from  $L_{p(x),\omega}(\Omega)$  to  $R$ .*

**Proof.** We argue by contradiction. Let  $\varphi \neq 0$  and  $\varphi \in [L_{p(x),\omega}(\Omega)]^*$ . Let  $\tilde{B}(0, t) = \Omega \cap B(0, t)$ , where  $0 < t < \infty$ .

Suppose that  $\varphi$  is linear continuous functional defined in  $L_{p(x),\omega}(\Omega)$ . Then we can find an  $f \in L_{p(x),\omega}(\Omega)$  such that  $\varphi(f) = 1$ . Now, the map  $t \mapsto f\chi_{\tilde{B}(0,t)}$  is continuous, since  $|f|^{p(x)}\omega(x)$  is integrable:

$$\int_{\tilde{B}(0,t_2)} |f(x)|^{p(x)} \omega(x) dx - \int_{\tilde{B}(0,t_1)} |f(x)|^{p(x)} \omega(x) dx = \int_{\Omega \cap B_{t_1,t_2}} |f(x)|^p \omega(x) dx, \quad \text{for } t_1 < t_2,$$

where  $B_{t_1,t_2} = \{x : t_1 \leq |y| < t_2\}$ . Thus we may choose  $t \in (t_1, \infty)$  such that  $\varphi(f\chi_{\tilde{B}(0,t)}) = \varphi(f\chi_{\Omega \setminus \tilde{B}(0,t)}) = \frac{1}{2}$ . Next, notice that  $g = f\chi_{\tilde{B}(0,t)}$  and  $h = f\chi_{\Omega \setminus \tilde{B}(0,t)}$  satisfy

$$\int_{\Omega} |f(x)|^{p(x)} \omega(x) dx = \int_{\tilde{B}(0,t)} |f(x)|^{p(x)} \omega(x) dx + \int_{\Omega \setminus \tilde{B}(0,t)} |f(x)|^{p(x)} \omega(x) dx = I_{p,\omega}(g) + I_{p,\omega}(h).$$

Thus, at least one of  $I_{p,\omega}(g)$  or  $I_{p,\omega}(h)$  is less than  $\frac{1}{2} I_{p,\omega}(f)$ . Let's say that  $I_{p,\omega}(g) \leq \frac{1}{2} I_{p,\omega}(f)$ . Then,  $f_1 = 2g$  satisfies

$$\varphi(f_1) = 1 \quad \text{and} \quad I_{p,\omega}(f_1) \leq 2^{\bar{p}} I_{p,\omega}(g) \leq 2^{\bar{p}-1} I_{p,\omega}(f).$$

By induction, we can find a sequence  $\{f_n\}_{n \geq 1}$  in  $L_{p(x),\omega}(\Omega)$  with

$$\varphi(f_n) = 1 \quad \text{and} \quad I_{p,\omega}(f_n) \leq 2^{n(\bar{p}-1)} I_{p,\omega}(f).$$

It is obvious that  $\bar{p} - 1 < 0$  and  $f_n \rightarrow 0$  in  $L_{p(x),\omega}(\Omega)$  while  $T(f_n) = 1$ . Thus,  $T = 0$  is the only continuous linear functional.

**Theorem 5.** *Let  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$  and  $\omega$  be a weight function defined on  $\Omega$  and  $0 < \omega(x) < \infty$  a.e.  $x \in \Omega$ . Then the spaces  $L_{p(x),\omega}(\Omega)$  is complete.*

**Proof.** Let  $\{f_n\}$ ,  $n \in N$  be a sequence in  $L_{p(x),\omega}(\Omega)$  such that

$$\|f_n - f_m\|_{p,\omega,\Omega} \rightarrow 0, \quad \text{for } n, m \rightarrow \infty.$$

From properties 1) implies that

$$\int_{\Omega} (|f_n - f_m| \omega(x))^{p(x)} dx \rightarrow 0, \quad \text{for } n, m \rightarrow \infty.$$

We choose the subsequence  $\{n_k\}$  such that

$$A = \sum_{k=1}^{\infty} \int_{\Omega} (|f_{n_{k+1}} - f_{n_k}| \omega(x))^{p(x)} dx < \infty.$$

Then for any  $\ell \in N$

$$\int_{\Omega} \left[ \sum_{k=1}^{\ell} (|f_{n_{k+1}} - f_{n_k}| \omega(x)) \right]^{p(x)} dx \leq \sum_{k=1}^{\ell} \int_{\Omega} (|f_{n_{k+1}} - f_{n_k}| \omega(x))^{p(x)} dx \leq A.$$

If  $\ell \rightarrow \infty$ , then by monotone convergence theorem

$$\int_{\Omega} \left[ \sum_{k=1}^{\infty} (|f_{n_{k+1}} - f_{n_k}| \omega(x)) \right]^{p(x)} dx \leq A.$$

Therefore,

$$\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \omega(x) < \infty, \quad a.e. \quad x \in \Omega.$$

Hence, by completeness of  $\mathbb{R}$ ,  $f_{n_k}$  converges a.e.  $x \in \Omega$ . We define a measurable function  $f$  by

$$f(x) = \begin{cases} \lim_{k \rightarrow \infty} f_{n_k}, & \text{for a.e. } x \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\int_{\Omega} (|f_n - f_m| \omega(x))^{p(x)} dx \rightarrow 0$ , for  $n, m \rightarrow \infty$ , then  $|f_n - f_m|^{p(x)} \rightarrow 0$ ,  $n, m \rightarrow \infty$ . Given  $\varepsilon > 0$  we can find  $N_{\varepsilon}$  so that  $n \geq N_{\varepsilon}$  implies

$$|\{x : |f_n(x) - f_m(x)|^{p(x)}\}| = \int_{\{x : |f_n(x) - f_m(x)|^{p(x)}\}} dx \leq \varepsilon, \quad \text{for } m \geq n.$$

In particular,  $|\{x : |f_n(x) - f_{n_k}(x)|^{p(x)}\}| \leq \varepsilon$ , for  $k \rightarrow \infty$ . Hence, by Fatou's lemma for  $n \geq N_{\varepsilon}$ , we have

$$\begin{aligned} |\{x : |f_n(x) - f(x)|^{p(x)}\}| &= \left| \liminf_{k \rightarrow \infty} \{x : |f_n(x) - f_{n_k}(x)|^{p(x)}\} \right| \\ &\leq \liminf_{k \rightarrow \infty} |\{x : |f_n(x) - f_{n_k}(x)|^{p(x)}\}| \leq \varepsilon. \end{aligned}$$

Hence  $f \in L_{p(x), \omega}(\Omega)$  and  $\int_{\Omega} |(f_n - f) \omega(x)|^{p(x)} dx \rightarrow 0$ , for  $n \rightarrow \infty$ .

This completes the proof of Theorem 5.

**Remark 4.** Note that from property 5) and Theorem 5 implies that the spaces  $L_{p(x), \omega}(\Omega)$  is ideal.

#### 4. Main results.

We consider the classical Hardy operator and it's dual operator defined as

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad H^*f(x) = \frac{1}{x} \int_x^\infty f(t) dt$$

where  $f$  is nonnegative function on  $(0, \infty)$ .

**Lemma 5.** *Let  $0 < \underline{p} \leq p_n \leq \bar{p} \leq 1$ ,  $p_n \geq p_{n+1}$  and  $\{x_n\}_{n \geq 1}$  be any non-negative sequence of real numbers such that  $x_n^{p_n} \geq x_{n+1}^{p_{n+1}}$  for any  $n \in \mathbb{N}$ .*

*Then*

$$\left( \sum_{n=1}^{\infty} x_n^{\frac{p_n}{p}} \right)^{\underline{p}} \leq \sum_{n=1}^{\infty} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] \leq \sum_{n=1}^{\infty} x_n^{p_n}. \quad (4.1)$$

**Proof.** First we proved that

$$\left( \sum_{n=1}^m x_n^{\frac{p_n}{p_m}} \right)^{p_m} \leq \sum_{n=1}^m x_n^{p_n} [n^{p_n} - (n-1)^{p_n}]. \quad (4.2)$$

We consider the function  $h(t) = \frac{(1+t)^q - 1}{t^q}$ , where  $t \geq 0$  and  $0 < q < 1$ . It is obvious that  $h'(t) = \frac{q[1 - (1+t)^{q-1}]}{t^{q+1}} \geq 0$  for all  $t \geq 0$ . In particular, the function  $h(t)$  is monotone increasing in the segment  $[0, B]$ . Therefore  $h(t) \leq h(B)$ , i.e.,

$$(1+t)^q \leq 1 + t^q [(B^{-1} + 1)^q - B^{-q}] \quad \text{for any } 0 \leq t \leq B. \quad (4.3)$$

Since  $x_1^{p_1} \geq x_2^{p_2}$ , then  $x_2 \leq x_1^{\frac{p_1}{p_2}}$ . Therefore taking  $t = \frac{x_2}{x_1^{\frac{p_1}{p_2}}}$ ,  $B = 1$  and  $q = p_2$  in (4.3), we

have

$$\left( x_1^{\frac{p_1}{p_2}} + x_2 \right)^{p_2} \leq x_1^{p_1} + x_2^{p_2} (2^{p_2} - 1). \quad (4.4)$$

It is obvious that the inequality (4.4) be inequality (4.2) for  $m = 2$ . By the condition of Lemma 2  $p_2 \geq p_3$  and so  $2^{p_3} \leq 2^{p_2}$ . Since  $x_3 \leq \frac{x_1^{\frac{p_1}{p_3}} + x_2^{\frac{p_2}{p_3}}}{2}$  from (4.3) and (4.4) for  $t = \frac{x_3}{\frac{x_1^{\frac{p_1}{p_3}} + x_2^{\frac{p_2}{p_3}}}{2}}$ ,  $B = \frac{1}{2}$  and  $q = p_3$ , we get

$$\begin{aligned} & \left( x_1^{\frac{p_1}{p_3}} + x_2^{\frac{p_2}{p_3}} + x_3 \right)^{p_3} \leq \left( x_1^{\frac{p_1}{p_3}} + x_2^{\frac{p_2}{p_3}} \right)^{p_3} + x_3^{p_3} (3^{p_3} - 2^{p_3}) \\ & \leq x_1^{p_1} + x_2^{p_2} (2^{p_3} - 1) + x_3^{p_3} (3^{p_3} - 2^{p_3}) \leq x_1^{p_1} + x_2^{p_2} (2^{p_2} - 1) + x_3^{p_3} (3^{p_3} - 2^{p_3}). \end{aligned}$$

The last inequality is (4.1) for  $m = 3$ . Clearly  $x_1^{\frac{p_1}{p_{m+1}}} + x_2^{\frac{p_2}{p_{m+1}}} + \dots + x_m^{\frac{p_m}{p_{m+1}}} + x_{m+1} \geq (m+1)x_{m+1}$ . Hence  $x_{m+1} \leq \frac{x_1^{\frac{p_1}{p_{m+1}}} + x_2^{\frac{p_2}{p_{m+1}}} + \dots + x_m^{\frac{p_m}{p_{m+1}}}}{m}$ . Therefore taking

$$t = \frac{x_{m+1}}{x_1^{\frac{p_1}{p_{m+1}}} + x_2^{\frac{p_2}{p_{m+1}}} + \dots + x_m^{\frac{p_m}{p_{m+1}}}}, B = \frac{1}{m} \text{ and } q = p_{m+1}$$

in (4.3), we have

$$\begin{aligned} \left( \sum_{n=1}^{m+1} x_n^{\frac{p_n}{p_{m+1}}} \right)^{p_{m+1}} &= \left( \sum_{n=1}^m x_n^{\frac{p_n}{p_{m+1}}} + x_{m+1} \right)^{p_{m+1}} \leq \\ &\left( \sum_{n=1}^m x_n^{\frac{p_n}{p_{m+1}}} \right)^{p_{m+1}} + x_{m+1}^{p_{m+1}} [(m+1)^{p_{m+1}} - m^{p_{m+1}}] \leq \\ &\sum_{n=1}^m x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] + x_{m+1}^{p_{m+1}} [(m+1)^{p_{m+1}} - m^{p_{m+1}}] = \\ &\sum_{n=1}^{m+1} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}]. \end{aligned}$$

By the induction principle the inequality (4.2) is proved for any  $m \in \mathbb{N}$ .

Since the sequence  $\{p_n\}_{n \geq 1}$  is decreasing, then  $\lim_{n \rightarrow \infty} p_n = \underline{p}$ . Therefore passing to the limit at  $m \rightarrow \infty$  in (4.2) we have the left part of inequality (4.1). By using the inequality  $n^{p_n} \leq (n-1)^{p_n} + 1$ , we have the right part of inequality (4.1).

The Lemma 2 is proved.

**Example 4.1.** Let  $x_n = \begin{cases} n^{-\frac{p}{2p_n}}, & \text{for } n = k^2 \\ 0, & \text{for } n \neq k^2, \end{cases}$  and  $\bar{p} < \frac{p+1}{2}$ .

It is obvious that the sequence  $\{x_n^{p_n}\}_{n \geq 1}$  isn't monotone and  $\sum_{n=1}^{\infty} x_n^{\frac{p}{2}} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$ .

On the other hand  $n^{p_n} - (n-1)^{p_n} \sim p_n n^{p_n-1} \sim n^{p_n-1}$  for  $n \rightarrow \infty$ . Therefore

$$\sum_{n=1}^{\infty} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] \sim \sum_{n=1}^{\infty} x_n^{p_n} n^{p_n-1} = \sum_{k=1}^{\infty} k^{-\underline{p}+2p_k-2} \leq \sum_{k=1}^{\infty} k^{2\bar{p}-\underline{p}-2}.$$

It is well known that the series  $\sum_{k=1}^{\infty} k^{2\bar{p}-\underline{p}-2}$  is converges if and only if  $\bar{p} < \frac{p+1}{2}$ . Thus for

$\bar{p} < \frac{p+1}{2}$  the inequality (3.1) isn't holds.

The example show that the condition of monotonicity of sequence  $\{x_n^{p_n}\}_{n \geq 1}$  is essential.

**Remark 5.** Note that Lemma 5 in the case  $p_1 = p_2 = \dots = p_n = \dots = p = \text{const}$  was proved in [5].

**Theorem 6.** Let  $x \in (0, \infty)$ ,  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$ ,  $r(x) = \frac{\underline{p}p(x)}{p(x) - \underline{p}}$  and  $f(x)$  are non-negative and decreasing function defined on  $(0, \infty)$ . Suppose  $\omega_1$  and  $\omega_2$  are weight functions defined on  $(0, \infty)$ .

Then for any  $f \in L_{p(x), \omega_1}(0, \infty)$  the inequality

$$\|Hf\|_{L_{q(\cdot), \omega_2}(0, \infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} c_{p,q} d_p \left\| \frac{t^{1/p'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t, \infty)}}{\omega_1} \right\|_{L_{r(\cdot)}(0, \infty)} \|f\|_{L_{p(\cdot), \omega_1}(0, \infty)},$$

where  $c_{p,q} = \left( \|\chi_{\Delta_1}\|_{L_\infty(0, \infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0, \infty)} + \underline{p} \left( \frac{1}{\underline{q}} - \frac{1}{\bar{q}} \right) \right) \left( \|\chi_{S_1}\|_{L_\infty(0, \infty)} + \|\chi_{S_2}\|_{L_\infty(0, \infty)} \right)$ ,  
 $S_1 = \{x \in (0, \infty) : p(x) = \underline{p}\}$ ,  $S_2 = (0, \infty) \setminus S_1$ , and  $d_p = \left( 1 + \frac{\bar{p} - \underline{p}}{\bar{p}} + \|\chi_{S_1}\|_{L_\infty(0, \infty)} \right)^{1/\underline{p}}$ .

**Proof.** Taking  $a = 0$ ,  $b = x$  and  $s = \underline{p}$  and apply Lemma 2 and property 5), we have

$$\begin{aligned} \|Hf\|_{L_{q(\cdot), \omega_2}(0, \infty)} &= \|\omega_2 Hf\|_{L_{q(\cdot)}(0, \infty)} = \left\| \frac{\omega_2}{x} \int_0^x f(t) dt \right\|_{L_{q(\cdot)}(0, \infty)} \\ &\leq \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{\omega_2(x)}{x} \left( \int_0^x f^{\underline{p}}(t) t^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0, \infty)}. \end{aligned}$$

Now applied Theorem 3, we get

$$\begin{aligned} &\left\| \frac{\omega_2(x)}{x} \left( \int_0^x f^{\underline{p}}(t) t^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0, \infty)} \\ &= \left\| \left( \int_0^\infty f^{\underline{p}}(t) \chi_{(0, x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0, \infty)} \\ &= \left\| \int_0^\infty f^{\underline{p}}(t) \chi_{(0, x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1} dt \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0, \infty)}^{1/\underline{p}} \end{aligned}$$

$$\begin{aligned}
&\leq c_{p,q} \left( \int_0^\infty \left\| f^{\underline{p}}(t) \chi_{(0,x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}} \\
&= c_{p,q} \left( \int_0^\infty f^{\underline{p}}(t) t^{\underline{p}-1} \left\| \chi_{(0,x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}} \\
&= c_{p,q} \left( \int_0^\infty f^{\underline{p}}(t) t^{\underline{p}-1} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)}^{\underline{p}} dt \right)^{1/\underline{p}} = c_{p,q} \left\| f t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)}.
\end{aligned}$$

Finally, apply Theorem 2, we get

$$\left\| f t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)} \leq d_p \left\| \frac{t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)}}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot), \omega_1}(0,\infty)}.$$

Thus

$$\|Hf\|_{L_{q(\cdot), \omega_2}(0,\infty)} \leq \underline{p}^{\frac{1}{2}} c_{p,q} d_p \left\| \frac{t^{1/\overline{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)}}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot), \omega_1}(0,\infty)}.$$

The Theorem 6 is proved.

**Theorem 7.** Let  $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$ ,  $r(x) = \frac{\underline{p}p(x)}{p(x) - \underline{p}}$  and  $f(x)$  are non-negative and increasing function defined on  $(0, \infty)$ . Suppose  $\omega_1$  and  $\omega_2$  are weight functions defined on  $(0, \infty)$ .

Then for any  $f \in L_{p(x), \omega_1}(0, \infty)$  the inequality

$$\|Hf\|_{L_{q(\cdot), \omega_2}(0,\infty)} \leq \underline{p}^{\frac{1}{2}} c_{p,q} d_p \left\| \left\| \frac{(x-t)^{1/\overline{p}'} \omega_2}{x} \right\|_{L_{q(\cdot)}(t,\infty)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot), \omega_1}(0,\infty)},$$

where  $c_{p,q}$  and  $d_p$  the constants in Theorem 6.

**Proof.** Taking  $a = 0$ ,  $b = x$  and  $s = \underline{p}$  and apply Lemma 3 and property 5), we have

$$\|Hf\|_{L_{q(\cdot), \omega_2}(0,\infty)} = \|\omega_2 Hf\|_{L_{q(\cdot)}(0,\infty)} = \left\| \frac{\omega_2}{x} \int_0^x f(t) dt \right\|_{L_{q(\cdot)}(0,\infty)}$$



$$\leq (\underline{p})^{1/\underline{p}} \left\| \frac{\omega_2(x)}{x} \left( \int_0^x f^{\underline{p}}(t) (x-t)^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)}.$$

Now applied Theorem 3, we get

$$\begin{aligned} & \left\| \frac{\omega_2(x)}{x} \left( \int_0^x f^{\underline{p}}(t) (x-t)^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)} \\ &= \left\| \left( \int_0^\infty f^{\underline{p}}(t) \chi_{(0,x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} (x-t)^{\underline{p}-1} dt \right)^{1/\underline{p}} \right\|_{L_{q(\cdot)}(0,\infty)} \\ &= \left\| \int_0^\infty f^{\underline{p}}(t) \chi_{(0,x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} (x-t)^{\underline{p}-1} dt \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)}^{1/\underline{p}} \\ &\leq c_p \left( \int_0^\infty \left\| f^{\underline{p}}(t) \chi_{(0,x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^{\underline{p}} (x-t)^{\underline{p}-1} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}} \\ &= c_p \left( \int_0^\infty f^{\underline{p}}(t) \left\| \chi_{(0,x)}(t) \left[ \frac{(x-t)^{1/\overline{p}'}}{x} \omega_2(x) \right]^{\underline{p}} \right\|_{L_{\frac{q(\cdot)}{\underline{p}}}(0,\infty)} dt \right)^{1/\underline{p}} \\ &= c_p \left( \int_0^\infty f^{\underline{p}}(t) \left\| \frac{(x-t)^{1/\overline{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(t,\infty)}^{\underline{p}} dt \right)^{1/\underline{p}} \\ &= c_p \left\| f \left\| \frac{(x-t)^{1/\overline{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)}. \end{aligned}$$

Finally, apply Theorem 2, we get

$$\begin{aligned} & \left\| f \left\| \frac{(x-t)^{1/\overline{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)} \\ &\leq \left\| \left\| \frac{(x-t)^{1/\overline{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(t,\infty)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0,\infty)} \|f\|_{L_{p(\cdot), \omega_1}(0,\infty)}. \end{aligned}$$

Thus

$$\|Hf\|_{L_{q(\cdot), \omega_2}(0, \infty)} \leq \underline{p}^{\frac{1}{2}} c_{p,q} d_p \left\| \left\| \frac{(x-t)^{1/\bar{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(t, \infty)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0, \infty)} \|f\|_{L_{p(\cdot), \omega_1}(0, \infty)}.$$

The Theorem 7 is proved.

For the dual operator  $H^*$  a theorem below is proved analogously.

**Theorem 8.** *Let  $x \in (0, \infty)$ ,  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$ ,  $r(x) = \frac{\underline{p}p(x)}{p(x) - \underline{p}}$  and  $f(x)$  are non-negative and decreasing function defined on  $(0, \infty)$ . Suppose  $\omega_1$  and  $\omega_2$  are weight functions defined on  $(0, \infty)$ .*

*Then for any  $f \in L_{p(x), \omega_1}(0, \infty)$  the inequality*

$$\|H^*f\|_{L_{q(\cdot), \omega_2}(0, \infty)} \leq \underline{p}^{\frac{1}{2}} c_{p,q} d_p \left\| \left\| \frac{(t-x)^{1/\bar{p}'}}{x} \omega_2 \right\|_{L_{q(\cdot)}(0, t)} \frac{1}{\omega_1} \right\|_{L_{r(\cdot)}(0, \infty)} \|f\|_{L_{p(\cdot), \omega_1}(0, \infty)},$$

where  $c_{p,q}$  and  $d_p$  the constants in Theorem 6.

**Remark 6.** *Note that Theorem 6, Theorem 7 and Theorem 8 in the case  $p(x) = q(x) = p = \text{const}$  and  $\omega_1(x) = \omega_2(x) = x^\alpha$  was proved in [6] (see also [5]). In the case  $1 \leq p(x) \leq q(x) \leq \bar{q} < \infty$  Hardy inequality is very much studied (see [2], [3] and etc.). In the constant exponent case  $1 \leq p \leq q \leq \bar{q} \leq \infty$  for detailed information we refer to [10]. Note that similar problem for Hardy maximal function was investigated in [9] and [11].*

**Example 4.2.** *Let  $x \in (0, \infty)$ ,  $0 < p(x) = p = \text{const} < 1$ ,  $q(x) = \begin{cases} \frac{1}{4}, & \text{for } 0 < x < 1 \\ \frac{1}{2}, & \text{for } x \geq 1, \end{cases}$   $0 < p \leq q(x)$  and  $p' = \frac{p}{p-1}$ . Suppose  $\omega_1(x) = x^\alpha$ ,  $\omega_2(x) = x^{\beta+1}$ ,  $\beta < -2$ ,  $\beta \neq -4$  and  $\beta + 2 + \frac{1}{p'} < \alpha < \min \left\{ \frac{1}{p'}; \beta + 4 + \frac{1}{p'} \right\}$ , where  $r(x) = \infty$ .*

*Then the pair  $(\omega_1, \omega_2)$  satisfies the condition of Theorem 6.*

**Example 4.3.** *Let  $x \in (0, \infty)$ ,  $0 < \underline{p} \leq p(x) \leq q(x) \leq \bar{q} < 1$  and  $\bar{p}' = \frac{\underline{p}}{\underline{p}-1}$ . Suppose  $\omega_1(x) = x^{1/\bar{p}'} \left\| \frac{\omega_2}{x} \right\|_{L_{q(\cdot)}(x, \infty)}$ . Then condition  $\|1\|_{L_{r(\cdot)}(0, \infty)} < \infty$  is guaranteed the satisfy of condition of Theorem 6. Note that by Definition 1 the condition  $\|1\|_{L_{r(\cdot)}(0, \infty)} < \infty$  is equivalent to*

$$\int_0^\infty \delta^{\frac{\underline{p}p(x)}{p(x)-\underline{p}}} dx < \infty,$$

where  $\delta \in (0, 1)$ . Then the pair  $(\omega_1, \omega_2)$  satisfies the condition of Theorem 6.

**Acknowledgement.** This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan EIF-2010-1(1)-40/06-1.

## References

- [1] R.A.Bandaliev, *On an inequality in Lebesgue space with mixed norm and with variable summability exponent*, Mat. Zametki, **3** (84)(2008), 323-333.(In Russian). English translation: Math. Notes, **3**(84)(2008), 303-313 (2008).
- [2] R.A.Bandaliev, *The boundedness of certain sublinear operator in the weighted variable Lebesgue spaces*, Czechoslovak Math. J. **60**(2), 327-337 (2010).
- [3] R.A.Bandaliev, *The boundedness of multidimensional Hardy operator in the weighted variable Lebesgue spaces*, Lithuanian Math. J. **50**(2010), no.3, 249-259.
- [4] R.A.Bandaliev, Z.V.Safarov, *Criteria of two-weighted inequalities for multidimensional Hardy type operators in weighted Musielak-Orlicz spaces and some applications*, Mathematische Nachrichten, 2012 (accepted).
- [5] R.A.Bandaliev, *Embedding between variable exponent Lebesgue spaces with measures*, Azerbaijan Journal of Math., **2**(1)(2012), 111-117.
- [6] R.A.Bandaliev and K. K. Omarova, *Two-weight norm inequalities for certain singular integrals*, Taiwanese Journal of Math., **2** (2012), 113-132.
- [7] J.Bergh, V.I.Burenkov, L.-E. Persson, *On some sharp reversed Hölder and Hardy-type inequalities*, Math. Nachr., 169 (1994), 19-29.
- [8] V.I. Burenkov, *On the exact constant in the Hardy inequality with  $0 < p < 1$  for monotone functions*, Trudy Matem. Inst. Steklov. 194 (1992), 58-62 (in Russian). English transl. in Proc. Steklov Inst. Math., 194, no. 4 (1993), 59-63.
- [9] L.Diening, P.Harjulehto, P.Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer Lecture Notes, v.2017, Springer-Verlag, Berlin, 2011.
- [10] O.Kováčik, J. Rákosník, *On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$* , Czechoslovak Math. J. (**41**)**116** (1991) 592-618.
- [11] A.K.Lerner, *On some questions related to the maximal operator on variable  $L^p$  spaces*, Trans. Amer. Math. Soc., **362**(2010), no. 8, 4229-4242.
- [12] V.G.Maz'ya, *Sobolev spaces*, (Springer-Verlag, Berlin, 1985).
- [13] B.Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **166**(1972).
- [14] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math.1034. Springer-Verlag, Berlin-Heidelberg-New York, 1983.

- [15] W. Orlicz, *Über konjugierte exponentenfolgen*, Studia Math. **3**(1931) 200-212.
- [16] K.R.Rajagopal, M. Růžička, *Mathematical modeling of electrorheological materials*, Cont. Mech. and Thermodyn., **13**(2001) 59-78.
- [17] S.G.Samko. *"Differentiation and integration of variable order and the spaces  $L^{p(x)}$ "* Proc.Inter.Conf "Operator theory for complex and hypercomplex analysis Mexico, 1994, *Contemp. Math.*, **212**(1998), 203-219.
- [18] I.I.Sharapudinov, *On a topology of the space  $L^{p(t)}([0, 1])$* , Matem. Zametki, **26**, 613-632 (1979) (in Russian): English translation: *Math. Notes*, **26**, 796-806 (1979).
- [19] Q.H.Zhang, *Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems*, Nonlinear Analysis TMA, (1)**70**(2009) 305-316.
- [20] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR.**50**(1986) 675-710. (In Russian). English transl.: Math. USSR, Izv., **29**(1987) 33-66.

DEPARTMENT OF MATHEMATICAL ANALYSIS, INSTITUTE OF MATHEMATICS  
AND MECHANICS OF NATIONAL ACADEMY OF SCIENCES OF AZERBAIJAN,  
Baku, Az 1141, B.Vahabzade str., 9  
*E-mail address*: bandaliyev.rovshan@math.ab.az